Comment. Math.
Univ. St. Pauli
XXVI—2, 1977

The Relation between Hein's Construction of Lie Algebras and the Construction of Lie Algabras from Symplectic Triple Systems

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(Received October 1, 1977)

Introduction. Let $\mathfrak{R}$ be a triple system, $\mathfrak{J}$ be a Jordan algebra with unit element and $\mathfrak{T}=\mathfrak{J}\oplus\mathfrak{R}$ be a triple system such that these satisfy some certain conditions (see §1, §2). From these, W. Hein, in [3], has constructed a Lie algebra of the form

$$\mathfrak{G}(\mathfrak{T})=L(\mathfrak{T},\mathfrak{T})\oplus\mathfrak{J}\oplus\bar{\mathfrak{J}}\oplus\mathfrak{R}\oplus\bar{\mathfrak{R}}\,,$$

that is a vector space direct sum with a suitable Lie product (cf. [3], p. 81), where $\bar{\mathfrak{J}}$ (resp. $\bar{\mathfrak{R}}$) is an isomorphic copy of $\mathfrak{J}$ (resp. $\mathfrak{R}$), (see §2 for the definition of $L(\mathfrak{T},\mathfrak{T})$).

On the other hand, from a symplectic triple system $\mathfrak{R}_1$ over a field K, K. Yamaguti and H. Asano, in [6], have constructed a Lie algebra $\mathfrak{G}(\mathfrak{R}_1)$ as follows. First, they construct the Lie triple system $\mathfrak{T}_1=\mathfrak{R}_1\oplus\bar{\mathfrak{R}}_1$ associated with $\mathfrak{R}_1$ where $\bar{\mathfrak{R}}_1$ is another copy of $\mathfrak{R}_1$ (cf. [6]). Let $\mathfrak{D}_0(\mathfrak{R}_1)$ (resp. $\mathfrak{D}_0(\mathfrak{T}_1)$) be the Lie algebra of inner derivations of $\mathfrak{R}_1$ (resp. $\mathfrak{T}_1$) (cf. §1). Using the fact that

$$\mathfrak{D}_0(\mathfrak{T}_1)=\mathfrak{D}_0(\mathfrak{R}_1)\oplus KU\oplus KV\oplus KW\,,$$

where U, V, W are the special inner derivations of $\mathfrak{T}_1$ (cf. [6]), we see that the standard enveloping Lie algebra of $\mathfrak{T}_1$ is of the form

$$\begin{aligned}\mathfrak{G}(\mathfrak{R}_1)&=\mathfrak{T}_1\oplus\mathfrak{D}_0(\mathfrak{T}_1)\\&=\mathfrak{D}_0(\mathfrak{R}_1)\oplus\mathfrak{R}_1\oplus\bar{\mathfrak{R}}_1\oplus KU\oplus KV\oplus KW\,.\end{aligned}$$

In the above situation, let $\mathfrak{J}$ be the one-dimensional Jordan algebra Ke. Then examining the relation between a triple system $\mathfrak{R}$ in Hein's definition and a symplectic triple system $\mathfrak{R}_1$ we can find that there is a $1:1$ correspondence between the class of $\mathfrak{R}$ and the class of $\mathfrak{R}_1$ (see Lemma 1 and 2) and that $\mathfrak{G}(\mathfrak{T})$ is isomorphic to $\mathfrak{G}(\mathfrak{R}_1)$ (see Theorem).

Throughout the paper, we assume that any vector space considered is finite-dimensional and the characteristic of the base field is different from 2 or 3.

§1. Let V be a vector space over a field K with a trilinear map $V\times V\times V\to V\colon (a, b, c)\mapsto[abc]\in V$. Then $(V, [\ \ ])$ is called a *triple system over K*.

Definition 1. (Yamaguti-Asano [6]). Let $\mathfrak{R}_1=(V, [\ \ ])$ be a triple system with a non-zero skew symmetric bilinear form α satisfying the following identities for any $a, b, c, d, g \in V$:

$$(S\ 1)\quad [abc]=[bac],$$
$$(S\ 2)\quad [abc]=[acb]=\alpha(a, c)b-\alpha(a, b)c+2\alpha(b, c)a,$$
$$(S\ 3)\quad [ab[cdg]]=[[abc]dg]+[c[abd]g]+[cd[abg]].$$

Then $\mathfrak{R}_1$ is called a *symplectic triple system.*

Definition 2 (Hein [3]). Let $\mathfrak{J}$ be a Jordan algebra with unit element e, $\mathfrak{R}=(V, \langle\ \ \rangle)$ be a triple system, (σ, V) be a special unital representation of $\mathfrak{J}$, and f be a non-zero skew symmetric bilinear mapping from $V\times V$ to $\mathfrak{J}$. Then a situation $\mathfrak{J}, \mathfrak{R}, f, \sigma$ is called *admissible* if the following identities hold for $a, b, c, d \in \mathfrak{R}$ and $x \in \mathfrak{J}$:

$$(F)\quad 2xf(a, b)=f(\sigma(x)a, b)+f(a, \sigma(x)b),$$
$$(L\ 1)\quad [L(a, b), L(c, d)]=L(L(a, b)c, d)+L(c, L(b, a)d),$$
$$(L\ 2)\quad \langle abc\rangle-\langle cba\rangle=\langle cab\rangle-\langle acb\rangle,$$
$$(V\ 1)\quad [\sigma(x), L(a, b)]=L(\sigma(x)a, b)-L(a, \sigma(x)b),$$
$$(V\ 2)\quad L(a, b)-L(b, a)=\sigma(f(a, b)),$$
$$(V\ 3)\quad f(\langle abc\rangle, d)+f(c, \langle abd\rangle)=f(a, \sigma(f(c, d))b),$$

where $L(a, b)$ is the endomorphism of V defined by $L(a, b)c:=\langle abc\rangle$.

In other words the pair $(\mathfrak{J}, \mathfrak{R})$ is called a *$\mathfrak{J}$-ternary algebra* with a skew form f (cf. [4]).

LEMMA 1. *Let a symplectic triple system $\mathfrak{R}_1=(V, [\ \ ])$ over a field K with a skew symmetric bilinear form α be given. Using unit element e of K we may regard K as the one-dimensional Jordan algebra Ke. Put $f(a, b):=-2\alpha(a, b)$ and also $\langle abc\rangle:=[abc]-\alpha(a, b)c$. Let σ be the linear mapping from K to $\mathrm{End}_k(V)$ such that $\sigma(e)=I$ (the identity endomorphism of V). Then a situation Ke, $\mathfrak{R}=(V, \langle\ \ \rangle)$, f, σ is admissible.*

LEMMA 2. *Conversely, let an admissible situation $\mathfrak{J}$, $\mathfrak{R}=(V, \langle\ \ \rangle)$, f, σ be given, in which $\mathfrak{J}$ is the one-dimensional Jordan algebra Ke. Put $\alpha(a, b)=-(1/2)f(a, b)$ and $[abc]:=\langle abc\rangle+\alpha(a, b)$. Then $\mathfrak{R}_1=(V, [\ \ ])$ with α is a symplectic triple system.*

For any triple system $\mathfrak{T}=(V, \{\ \ \})$, an endomorphism D of V is called a derivation of $\mathfrak{T}$ if $D\{abc\}=\{(Da)bc\}+\{a(Db)c\}+\{ab(Dc)\}$. Let $\mathfrak{D}(\mathfrak{T})$ denote the set of all derivations of $\mathfrak{T}$ then $\mathfrak{D}(\mathfrak{T})$ becomes a Lie algebra. Let $L(\mathfrak{T}, \mathfrak{T})$ be the vector space spanned by all $L(a, b)$ for $a, b \in \mathfrak{T}$ then the set $\mathfrak{D}_0(\mathfrak{T})=L(\mathfrak{T}, \mathfrak{T})\cap\mathfrak{D}(\mathfrak{T})$ is called the *Lie algebra of inner derivations* of $\mathfrak{T}$. From now on we assume that $\mathfrak{R}$ corresponds to $\mathfrak{R}_1$ as in Lemma 1 and Lemma 2.

LEMMA 3. (i) *For any* $D\in\mathfrak{D}(\mathfrak{K}_1)$, $\alpha(Da, b)+\alpha(a, Db)=0$.
(ii) *For any* $D\in\mathfrak{D}(\mathfrak{K})$, $\alpha(Da, b)+\alpha(a, Db)=0$.

Proof of (i). Applying $D\in\mathfrak{D}(k)$ to the identity (S 2) we obtain

$$[(Da)bc]+[a(Db)c]+[ab(Dc)]-[(Da)cb]-[a(Dc)b]-[ac(Db)]$$
$$=\alpha(a, c)Db-\alpha(a, b)Dc+2\alpha(b, c)Da\ .$$

Using (S 2) and putting $c=a$, we obtain

$$\alpha(b, Da)+\alpha(Db, a)=0 \qquad \text{for any} \quad a, b\in\mathfrak{K}_1\ .$$

Proof of (ii). From (L 2), (S 1) and the relation between $\mathfrak{K}$ and $\mathfrak{K}_1$, we obtain

(1) $$\langle abc\rangle-\langle cba\rangle=2\alpha(a, c)b\ .$$

Applying $D\in\mathfrak{D}(\mathfrak{K})$ to the above identity (1), we obtain

$$\langle (Da)bc\rangle+\langle a(Db)c\rangle+\langle ab(Dc)\rangle-\langle (Dc)ba\rangle-\langle c(Db)a\rangle-\langle cb(Da)\rangle$$
$$=2\alpha(a, c)Db\ .$$

Using (1) we have $\alpha(Da, c)+\alpha(a, Dc)=0$ for any $a, c\in\mathfrak{D}(\mathfrak{K})$.
From the relation between $\mathfrak{K}$ and $\mathfrak{K}_1$ we get

$$D\langle abc\rangle-\langle (Da)bc\rangle-\langle a(Db)c\rangle-\langle ab(Dc)\rangle$$
$$=D[abc]-[(Da)bc]-[a(Db)c]-[ab(Dc)]+\alpha(Da, b)c+\alpha(a, Db)c$$

for any endomorphism D of V. Using Lemma 3 we have the following

LEMMA 4. $\mathfrak{D}(\mathfrak{K})=\mathfrak{D}(\mathfrak{K}_1)$.

From now on, let $L(a, b)$ (resp. $L_1(a, b)$) denote the endomorphism of V defined by $L(a, b)c:=\langle abc\rangle$ (resp. $L_1(a, b)c:=[abc]$). Then we have

(2) $$L(a, b)=L_1(a, b)-\alpha(a, b)I\ .$$

For $a, b\in V$, we define t e endomorphism $D(a, b)$ of V by

(3) $$D(a, b):=-\frac{1}{2}(L(a, b)+L(b, a))\ ,$$

then using (S 1) and (2) we obtain

(4) $$D(a, b)=-L_1(a, b)$$

for $a, b\in V$. Let $D(\mathfrak{K}, \mathfrak{K})$ be the vector space spanned by all $D(a, b)$ for $a, b\in\mathfrak{K}$. Then (3) and (4) imply $\mathfrak{D}_0(\mathfrak{K}_1)=D(\mathfrak{K}, \mathfrak{K})\subset\mathfrak{D}_0(\mathfrak{K})$. Using the fact that α is not a zero form, we have the following

LEMMA 5. (i) $L(\mathfrak{K}, \mathfrak{K})=D(\mathfrak{K}, \mathfrak{K})\oplus KI$.
(ii) $D(\mathfrak{K}, \mathfrak{K})=\mathfrak{D}_0(\mathfrak{K})$.
(iii) $\mathfrak{D}_0(\mathfrak{K})=\mathfrak{D}_0(\mathfrak{K}_1)$.

Proof. From (2) and (S 1) we obtain

$$\alpha(a, b)I = -\frac{1}{2}(L(a, b) - L(b, a)) . \tag{5}$$

Using (3) and (5) we get $L(a, b) = -D(a, b) - \alpha(a, b)I$. Hence $L(\mathfrak{R}, \mathfrak{R})$ is the vector space sum of $D(\mathfrak{R}, \mathfrak{R})$ and KI. Since I is not a derivation, this sum is direct and then (ii) and (iii) are also clear.

§2. Let an admissible situation $\mathfrak{J}$, $\mathfrak{R}$, f, σ be given. In [3], W. Hein has made the triple system $\mathfrak{T}$, which is the vector space direct sum of $\mathfrak{J}$ and $\mathfrak{R}$, as follows. For $x, y, z \in \mathfrak{J}$ and $a, b, c \in \mathfrak{R}$ we put $\{xyz\} := (xy)z + x(yz) - y(xz)$, denote the product in $\mathfrak{R}$ by $\langle\ \ \rangle$ and define

$$\{xy(z+c)\} := \{xyz\} - \frac{1}{2}\sigma(y)\sigma(x)c ,$$
$$\{ab(z+c)\} := -f(\sigma(x)a, b) + \langle abc \rangle ,$$
$$\{xb(z+c)\} := \{ay(z+c)\} := 0 .$$

Put $L(x) := L(x, e)$ for $x \in \mathfrak{J}$, where $L(u, v)w := \{uvw\}$ for $u, v, w \in \mathfrak{T}$. Then we obtain $L(x) = xy$ and $L(x)a = -(1/2)\sigma(x)a$ for $x, y \in \mathfrak{J}$ and $a \in \mathfrak{R}$. Moreover,

$$L(x, y) = L(xy) + [L(x), L(y)] , \tag{6}$$

$$L(a, b) - L(b, a) = -2L(f(a, b)) , \tag{7}$$

for $x, y \in \mathfrak{J}$ and $a, b \in \mathfrak{R}$. Next we define a bilinear mapping $D' : \mathfrak{T} \times \mathfrak{T} \to \mathrm{End}_K(\mathfrak{T})$ by

$$D'(x, y) := \frac{1}{2}(L(x, y) - L(y, x)) ,$$

$$D'(a, b) := -\frac{1}{2}(L(a, b) + L(b, a)) ,$$

$$D'(x, a) := D'(a, x) := 0$$

for $x, y \in \mathfrak{J}$ and $a, b \in \mathfrak{R}$. Using (6) and (7) we obtain

$$L(x, y) = L(xy) + D'(x, y) ,$$
$$L(a, b) = -L(f(a, b)) - D'(a, b)$$

for $x, y \in \mathfrak{J}$ and $a, b \in \mathfrak{R}$. Hence $L(\mathfrak{T}, \mathfrak{T})$ is the vector space sum of $L(\mathfrak{J})$ and $D'(\mathfrak{T}, \mathfrak{T})$, where $D'(\mathfrak{T}, \mathfrak{T})$ is the Lie algebra of inner derivations of $\mathfrak{T}$ (cf. [3] p. 86). This sum is direct since $D'(x+a, y+b)e = 0$ for any $x, y \in \mathfrak{J}$ and $a, b \in \mathfrak{R}$.

In case $\mathfrak{J} = Ke$, it is clear that $\mathfrak{T} = Ke \oplus \mathfrak{R}$ and $L(\mathfrak{T}, \mathfrak{T}) = KL(e) + D'(\mathfrak{R}, \mathfrak{R})$. Since any derivation $D' \in D'(\mathfrak{R}, \mathfrak{R})$ maps Ke into 0 and $\mathfrak{R}$ into $\mathfrak{R}$, $D'|\mathfrak{R}$ which is the restriction of D' to $\mathfrak{R}$ belongs to $D(\mathfrak{R}, \mathfrak{R})$. Then the restriction map $D' \to D'|\mathfrak{R}$ is a Lie isomorphism of $D'(\mathfrak{R}, \mathfrak{R})$ onto

$D(\mathfrak{R}, \mathfrak{R})$. Since $-2L(e)$ is the identity endomorphism on $\mathfrak{R}$ and maps Ke into Ke, the linear mapping φ defined by $\varphi(D')=D'|\mathfrak{R}$ and $\varphi(-2L(e))=I$ is a Lie isomorphism of $D'(\mathfrak{R}, \mathfrak{R})\oplus KL(e)$ onto $D(\mathfrak{R}, \mathfrak{R})\oplus KI$.

THEOREM. *The Lie algebra constructed from an admissible situation Ke, $\mathfrak{R}$, f, σ is as follows:*

$$\mathfrak{G}(\mathfrak{T})=Ke\oplus\bar{\mathfrak{R}}\oplus KL(e)\oplus D'(\mathfrak{R}, \mathfrak{R})\oplus\mathfrak{R}\oplus K\bar{e} .$$

On the other hand, the Lie algebra constructed from a symplectic triple system is as follows:

$$\mathfrak{G}(\mathfrak{R}_1)=KW\oplus\bar{\mathfrak{R}}_1\oplus KU\oplus\mathfrak{D}_0(\mathfrak{R}_1)\oplus\mathfrak{R}_1\oplus KV .$$

Then $\mathfrak{G}(\mathfrak{T})$ is isomorphic to $\mathfrak{G}(\mathfrak{R}_1)$ under the linear mapping φ defined by

$$\varphi(e)=W , \qquad \varphi(\bar{e})=V , \qquad \varphi(-2L(e))=U ,$$

φ is the restriction map of $D'(\mathfrak{R}, \mathfrak{R})$ onto $\mathfrak{D}_0(\mathfrak{R}_1)$,
φ is the identity mapping from $\mathfrak{R}$ onto $\mathfrak{R}_1$ which induces the linear mapping from $\bar{\mathfrak{R}}$ onto $\bar{\mathfrak{R}}_1$.

Remark. To verify that φ is a Lie homomorphism, compare both the Lie products in [2] and [3].

Since it is shown in [2] that each simple Lie algebra (except A_1) over an algebraically closed field of characteristic 0 can be constructed from a symplectic triple system, we have the following

COROLLARY. *Each simple Lie algebra (except A_1) over an algebraically closed field of characteristic 0 can be constructed from an admissible situation.*

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